SOLUTIONS OF PROBLEMS IN NO 3.

Solutions of problems in No. 3 have been received as follows:

From P. E. Chase, 69, 70, 72 & 73; G. M. Day, 70, 71, 72 & 74; Th. L. De Land, 72; Cadet E. S. Farrow, 69, 70, 71, 72, 73 & 74; H. Gunder, 69, 70, 71, 72 & 74; Wm. Hoover, 69, 71 & 72; Phil. Hoglan, 72; Prof. W. W. Johnson, 74; Christine Ladd, 73; F. P. Matz, 72; E. P. Norton, 69; O. D. Oathout, 70, 72 & 73; Judge Josiah Scott, 73; Prof. J. Scheffer, 69, 70, 71, 72, 73 & 74; E. B. Seitz, 69, 70, 71, 72, 73 & 74.

69.—"Given $x^2 + y^2 = 793$, . . . (1) $\sqrt[3]{(xy^2)} + \sqrt[3]{(x^2y)} = 30$, . . . (2) to find x and y by quadratics."

SOLUTION BY PROF. P. E. CHASE, PHILAD., PA.

$$x^2 \pm 2xy + y^2 = 793 \pm 2xy = \square,$$
$$\sqrt[3]{(xy)}(\sqrt[3]{x} + \sqrt[3]{y}) = 30 = 6(3+2), \quad xy = 216, \quad x = 27, \quad y = 8.$$

SOLUTION BY E. P. NORTON, ALLEN, MICHIGAN.

Cube (2), $xy^2 + 3xy[\sqrt[3]{(xy^2)}+\sqrt[3]{(x^2y)}] + x^2y = 27000$. . (3). Substituting in (3), the value of $\sqrt[3]{(xy^2)} + \sqrt[3]{(x^2y)}$, and we have $xy(x + y + 90) = 27000$. . . .(4). Put $x + y = s$ and $xy = p$, then by making the proper substitutions in (2) and (4), we have $s^2 - 2p = 793$, (5), and $p(s + 90) = 27000$, (6). Eliminate p from (5) and (6) and $s^3+90s^2-793s=125370$. (7) Let $s = z - 30$, then by substitution, $z^3 - 3493z = 47580$, (8). Multiply (8) by z, complete the square by adding $4225z^2 + (366)^2$ to each member of the equation, reduce, and we have $z = 65$, $s = 35$ and $p = 216$. Whence we readily find $x = 27$ and $y = 8$.

70.—"Given $y^2 + z^2 = 2500$,... (1), $x^2 = 1600[130 - (x + t)]^2$, . . . (2) $xy = t[130 - (x + t)]$, . . (3), $t^2 = y^2 + (z - 40)^2$, . . .(4), to find x, y and z."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.

Extract square root of (2) and divide by (3), this gives $t = 40y$. . . . . (5) Eliminate t and y from (4) by means of (1) and (5) and we have a quadratic from which we find $z = 49.99$. The other quantities are readily found by substitution.

[Subsequent to the publication of this question Mr. Matz wrote to us that he had *intended* to state eqn. (2) thus; $x^2 = 1600 + [130 - (x + t)]^2$, and that he had written to several of our contributors correcting the error.

Accordingly we have received several solutions of the question as modified by Mr. Matz by substituting in eqn. (2) after 1600 the sign $+$ instead of $\times$. We subjoin a solution, of the question thus modified, by E. B. Seitz].

Solution. Multiply (2) by t^2, square (3), and we have, by subtraction, $x^2(t^2-y^2)=1600t^2$... (5). Multiplying (4) by x^2, subtracting it from (5), and extracting the square root, we have $x(z-40)=\pm 40t$. . . . . . . (6)
Adding (1) and (4), we have $t^2=4100-80z$. . . . (7)

From (7), (6) and (2) we have, respectively,

$$z=\frac{4100-t^2}{80},\ x=\pm\frac{40t}{z-40}=\pm\frac{3200t}{900-t^2},\ x=\frac{t^2-260t+18500}{260-2t}.$$

$$\therefore\ \frac{t^2-260t+18500}{260-2t}=\pm\frac{3200t}{900-t^2}.$$

$$\therefore\quad t^4-260t^3+11200t^2+1066000t-16650000=0; \ldots\ldots (8)$$

$$\text{or}\quad t^4-260t^3+24000t^2-598000t-16650000=0. \ldots\ldots (9)$$

Solving (8) by Descartes' method we find $t=14.16625$, or -51.23628. $\therefore z=48.74146$, or 18.43554; $x=40t\div(z-40)=64.82325$, or 95.03838; $y=11.14763$, or -46.47723. From (9) we find $t=90$, or -15.87797; $\therefore z=-50$, or 48.09863; $x=40t\div(40-z)=40$, or 78.42304; $y=0$, or -13.65731.

71.—"Given the sides a, b, c of a spherical triangle ABC to find the radii R, r, of the circumscribed and inscribed circles."

[Several of our contributors write that this question is solved in Chauvenet's Trig., and all the solutions sent give for answers

$$\tan R=\frac{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{\sqrt{(\sin s\sin(s-a)\sin(s-b)\sin(s-c)}},$$

$$\tan r=\sqrt{\left(\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}\right)};$$

where $s=\frac{1}{2}(a+b+c)$].

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72. —"Find the maximum cylinder that can be cut from a given oblate spheroid whose semi-axes are a and b."

SOLUTION BY WM. HOOVER, BELLEFONTAINE, OHIO.

Let x be the radius and $2y$ the perpendicular of the cylinder. Then its volume is $u=2\pi x^2y=$ a max. . . . . (1)
From the ellipse $a^2y^2+b^2x^2=a^2b^2$. . . . . (2)
Substituting for x^2 in (1) its value from (2), differentiating and equating to zero we find $y=\frac{1}{3}b\sqrt{3}$. Substitute this value of y in (2) aud we get

$$x^2=\tfrac{2}{3}a^2.\qquad \therefore u=\tfrac{4}{9}\pi a^2b\sqrt{3}.$$

73.—"Find the whole number of sets of three integers having a constant sum."

SOLUTION BY CHRISTINE LADD, CHELSEA, MASS.

Let $n =$ the constant sum. Writing down the sets which contain 1,

(*a*)	(*b*)	(*c*)
1	1	$n - 2$
1	2	$n - 3$
1	3	$n - 4$
. . . . .	. . . . .	. . . . .

it is readily seen that they begin to repeat themselves when $(b) = (c)$ or $(b) + 1 = (c)$ according as n is odd or even. Put $n = 2p + q$, in which p is integral and $q < 2$; i. e. $q = 1$ if n is odd and $q = 0$ if n is even.

Then $$(b) = n - 1 - (b) = \frac{n-1}{2} \text{ when } q = 1,$$

or $$(b) = n - 1 - (b) - 1 = \frac{n-1-1}{2} \text{ when } q = 0;$$

$$(b) = p + q - 1 \text{ in both cases.}$$

But the maximum value of (b) is also the number of sets containing 1.

Take next the sets which contain 2. $2 + 1 + n - 3$ has already occurred so we begin with

(*a*)	(*b*)	(*c*)
2	2	$n - 4$
2	3	$n - 7$
2	4	$n - 8$
. . . . .	. . . . .	. . . . .

which will continue until

$$(b) = n - 2 - (b) = \frac{n-2}{2} \text{ when } q = 0,$$

or $$(b) = n - 2 - (b) - 1 = \frac{n-2-1}{2} \text{ when } q = 1.$$

$(b) = p - 1$ in both cases, which is one more than the number of sets containing 2, $= p - 1 - 1$.

(*a*)	(*b*)	(*c*)
3	3	$n - 6$
3	4	$n - 7$
3	5	$n - 8$
. . . . .	. . . . .	. . . . .

until $$(b) = \frac{n-3}{2} \; (q = 1), \text{ or } (b) = \frac{n-3-1}{2} \; (q = 0).$$

$(b) = p + q - 2$ in both cases, and $p + q - 2 - 2 =$ number of sets containing 3.

Collecting the results we are able to deduce the law of formation.

(1) $p + q - 1$	(2) $p - 1 - 1$
(3) $p + q - 2 - 2$	(4) $p - 2 - 2$
(5) $p + q - 3 - 4$	(6) $p - 3 - 5$
(7) $p + q - 4 - 6$	(8) $p - 4 - 7$
(9) $p + q - 5 - 8$	(d) . . . .

Collecting similar terms and adding,

$$(A) \qquad\qquad (B)$$

$$p + q - 1 - 3(0, 1, 2, \ldots.) \qquad p - 2 - 3(0, 1, 2, \ldots.)$$

in which it remains to find the limit of the series $(0, 1, 2, \ldots.)$.

Put $n = 3(2k + m) + l$ in which k, m, l, are integral, $l < 3$, $m < 2$. (d) will increase up to $2k + m$. Of these variations of sets $k + m$ will be in (A) and k in (B). Hence we shall have,

$$(A) + (B) = (k + m)(p + q - 1) + k(p - 2) - 3[0, 1, 2, \ldots (k + m - 1)] - 3[0, 1, 2, \ldots (k - 1)].$$

74.—"Find the equation of the locus of the middle point of a chord to the hyperbola $x^2 - y^2 = 2a^2$, the chord being of constant length and equal to seven times the transverse axis."

SOLUTION BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Combining the equation of the straight line $y = mx + b$ with that of the hyperbola we find for the intersections

$$x = \frac{bm \pm \sqrt{[2a^2(1 - m^2) + b^2]}}{1 - m^2} \text{ and } y = \frac{b \pm m\sqrt{[2a^2(1 - m^2) + b^2]}}{1 - m^2};$$

hence the coordinates of the middle point are

$$x = \frac{bm}{1 - m^2} \text{ and } y = \frac{b}{1 - m^2}$$

and the sum of the squares of the radical parts equals the square of half the chord equals the square of seven times the transverse semi-axis; that is

$$(1 + m^2)\left[\frac{2a^2}{1 - m^2} + \frac{b^2}{(1 - m^2)^2}\right] = 98a^2.$$

Eliminating b & c $(1 + m^2)(2a^2 + y^2 - m^2y^2) = 98a^2(1 - m^2)$,

but $m = \frac{x}{y}$ hence $\left(1 + \frac{x^2}{y^2}\right)\left(2a^2 + y^2 - x^2\right) = 98a^2\left(1 - \frac{x^2}{y^2}\right)$

or $$(y^2 + x^2)(2a^2 + y^2 - x^2) = 98a^2(y^2 - x^2)$$

or $$y^4 - x^4 + 100a^2x^2 - 96a^2y^2 = 0.$$

Note. This is the curve known as "*la courbe du diable*" which is given by its equation as an example in all the books on curve-tracing. If we

change the coefficient of a^2y^2 from 96 to 100 the equation represents the straight lines $y=\pm x$ and the circle $x^2+y^2=100$, to which system of lines the curve is therefore closely asymptotic. I conjecture that the curve was originated in this way and has not heretofore been known as a geometrical locus. Can any of your readers throw any light on the history of this curious curve and its startling title?

Answer to Prof. Hall's Query, by Prof. W. W. Johnson.

Prof. Cayley proposed the question "Find the number of regions into which infinite space is divided by n planes" in the Smith Prize Examination Feb. 3rd, 1874, and published in the Mathematical Messenger for march 1874, "Solutions and remarks" on the paper of that day. He says he intended the questson for a problem, as the result, though a known, is not a generally known one. His solution is substantially as follows: Consider the analogous problem for lines in a plane. An additional line adds to the number of regions one for every part into which it is itself divided by the other lines. Hence, 1, 2, 3, 4 &c. lines divide a plane into 2, $2+2(=4)$, $4+3(=7)$, $7+4(=11)$ &c. regions; the general term being $\frac{1}{2}(n^2+n+2)$. In like manner an additional plane adds to the number of regions in space one for every region in to which it is itself divided by the other planes. Hence 1, 2, 3, 4, &c. planes divide space into 2, $2+2(=4)$, $4+4(=8)$, $8+7(=15)$, $15+11(=26)$ &c. regions; the general term being $\frac{1}{6}(n^3+5n+6)$.

[Mr. G. W. Hill obtains the same result as answer to Prof. Hall's Query and by analogous reasoning, employing however in his investigation the Calculus of Finite Differences.

It will be observed that the question as proposed by Prof. Cayley is not identical with that proposed by Prof. Hall; as Prof. Cayley requests the number of regions into which infinite space *is* divided by n planes, whereas Prof. Hall asks, "Into how many parts *can* n planes divide space."

That the answer given is not *necessarily* the answer to Prof. Cayley's question follows from the fact that nothing in Prof. Cayley's announcement of the question precludes the possibility (theoretically at least) of some or all of the planes being parallel, in which case the answer would obviously not be correct: If drawn at random, however, the probability of such a contingency is infinitely small.—Ed.]

Answer to President Tappan's Query, by Prof. J. Scheffer.

It is. The French mathematician *Fermat* who published quite a number of theorems in regard to prime numbers, erroneously asserted that all the